NST IB: Mathematical Methods I Sheet 8: Power series solutions to ODEs

Corrections to `ghm22@cam.ac.uk`. *Star* ($\star$) *indicates a harder question.*

1 Given that $y = x$ is one solution of

$$(1-x^2)y'' - 2xy' + 2y = 0,$$

find the general solution.

2 Find two linearly independent power series solutions, about $x = 0$ of

$$y'' - xy = 0.$$

Discuss the convergence of the series.

3 Find power series solutions, about $x = 0$, of the differential equation

$$4xy'' + 2y' + y = 0.$$

Sum the series to show that two solutions are $\cos\sqrt{x}$ and $\sin\sqrt{x}$. Are they linearly independent?

4 Identify the singular points of the equation,

$$(2z + z^3)y'' - y' - 6zy = 0,$$

and determine their nature. Find two linearly independent solutions as power series about $z = 0$. In particular, determine the indicial equation, the recurrence relation, and the radius of convergence of your solutions.

5 Show that $x = 0$ is an irregular singular point of the differential equation

$$x^3y'' + y = 0.$$

By attempting a solution of the form $y = x^\sigma \sum_{n=0}^{\infty} a_n x^n$, demonstrate why the method of Frobenius fails to yield a valid power series solution in this case.

6 Use the method of Frobenius to find the general solution of

$$xy'' + 2y' + xy = 0$$

about $x = 0$. Show that although the roots of the indicial equation differ by an integer, a second solution can be found without introducing a logarithmic term. Express the general solution in terms of elementary functions.

7 Find power series solutions, about $x = 0$ of

$$x(x-1)y'' + 3xy' + y = 0.$$

8 Find power series solutions, about $x = 0$ of

$$xy'' + (c-x)y' - ay = 0 \quad (c \notin \mathbb{Z}).$$

9 The Schrödinger equation for a harmonic oscillator of energy E can be written as

$$-\hbar^2\Psi''(x) + mkx^2\Psi(x) = 2mE\Psi(x)$$

for some constants m and k. Show how to transform this into the equation

$$w'' + (\lambda - \xi^2)w = 0$$

for a function $w(\xi)$. Write $w = y\exp\bigl(-\xi^2/2\bigr)$ to obtain Hermite's equation for $y(\xi)$:

$$y'' - 2\xi y' + (\lambda - 1)y = 0.$$

Find two power series solutions (about $\xi = 0$) and show that both are convergent for all finite ξ (they behave like $\exp\xi^2$). Show that polynomial solutions are possible for particular values of λ. Given that $|\Psi|^2$ is the probability density and should be integrable, explain why these are the only acceptable solutions and hence obtain a formula for the energy levels of the harmonic oscillator.

10 Find two linearly independent power series solutions, about $x = 0$, of Chebyshev's equation

$$(1 - x^2)y'' - xy' + \alpha^2 y = 0,$$

where α is a constant. Show that if $\alpha = n$ (a non-negative integer), one of the series terminates to form a polynomial of degree n. Find the first three such polynomials.

11 Locate and classify all the singular points in the complex plane of the differential equation

$$(x^2 + 4)y'' + 2xy' - 12y = 0.$$

Without solving the equation, determine the minimum radius of convergence of a power series solution about $x = 0$. Find the general solution in powers of x.

12 Find solutions of Legendre's equation

$$(1 - x^2)y'' - 2xy' + l(l + 1)y = 0$$

in the form of inverse power series

$$y = \sum_{n=0}^{\infty} a_n x^{\sigma - n}$$

Show that either $\sigma = l$ or $\sigma = -(l + 1)$ and that one of the solutions is a polynomial in x when l is a positive integer.