Corrections to `ghm22@cam.ac.uk`*. Star ($\star$) indicates a harder question.*

This course covers a bit of complex *analysis* as a precursor to complex *methods* (contour integrals) which come in Lent term.

Health warning: this is a difficult topic, and possibly whizzed through a bit in lectures. I've added a lot of additional questions here so you're comfortable with the full range of questions that might come up in the exam.

1 Define the radius of convergence of a power series of a complex variable z. Prove that the radius of convergence exists. Find the radii of convergence of the three series

$$\sum_{n=1}^{\infty} \frac{z^n}{n}, \quad \sum_{n=0}^{\infty} (\cosh n) z^n, \quad \text{and} \quad \sum_{n=0}^{\infty} [2^n + (-1)^n] z^n.$$

Show by example that a power series may or may not converge on its circle of convergence. Hence give an example of a series that is convergent but not absolutely convergent.

2 Calculate the Taylor series of the function $f(z) = \log(1 - z)$ about $z = 0$ and determine its radius of convergence. Now calculate the Taylor series for $f(z)$ about $z = i$ and determine the new radius of convergence. Comment.

3 Find the first three terms of the Taylor series expansion of the function:

$$f(z) = \frac{z}{e^z - 1}$$

about $z = 0$. Show that the singularity at $z = 0$ is removable.

By identifying the positions of all other singularities of $f(z)$ in the complex plane, determine the exact radius of convergence of this power series.

4 Where are the zeros and singularities of the following complex functions? Give the orders of the zeros, and classify the singularities.

$$\frac{(z-i)^2}{z+1}, \quad \frac{1}{1+z} - \frac{1}{1-z}, \quad \frac{1}{z^2+i}, \quad \sec^2 \pi z, \quad \sin z^{-2}, \quad \sinh \frac{z}{z^2-1}, \quad \frac{\tanh z}{z}.$$

5 To examine the behaviour of a complex function $f(z)$ at infinity, we make the substitution $z = 1/\zeta$ and investigate the behaviour of the function $g(\zeta) = f(1/\zeta)$ at $\zeta = 0$.

Classify the behaviour at $z = \infty$ (analytic, pole of a certain order, or essential singularity) for the following functions:

$$f(z) = z^4 - 3z^2 + 1, \quad f(z) = \frac{z^2+1}{2z^2 - z}, \quad f(z) = e^z, \quad f(z) = \sin\left(\frac{1}{z}\right).$$

6 What is meant by an *isolated singularity* of a complex function $f(z)$?

Consider the function:

$$f(z) = \frac{1}{\sin(\pi/z)}.$$

Find all the singularities of $f(z)$ in the finite complex plane. Show that the singularities at $z_n = 1/n$ (where $n \in \mathbb{Z}^+$) are isolated simple poles. Explain why the singularity at $z = 0$ is *not* an isolated singularity. Can $f(z)$ be expanded in a Laurent series valid in a punctured disk $0 < |z| < R$?

7 State the definition of complex differentiability for a function $f(z)$ at a point z_0.

By considering paths parallel to the real and imaginary axes, derive the Cauchy–Riemann equations for $f(z) = u(x, y) + iv(x, y)$, where $z = x + iy$.

Show that the function $f(z) = \bar{z}$ (the complex conjugate of z) is nowhere complex differentiable. Show also that $f(z) = |z|^2$ is complex differentiable at $z = 0$, but is not analytic there.

8 Show that if a function $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain $\mathcal{D}$, then both $u(x, y)$ and $v(x, y)$ satisfy Laplace's equation:

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0.$$

9 Show that the function on the plane defined by

$$\Phi(x, y) = \operatorname{Im}\left\{\frac{2}{\pi} \log \tanh(x + iy)\right\}$$

satisfies Laplace's equation for $x > 0$. Show also that $\Phi = 0$ on both $y = 0$ and on $y = \pi/2$ and that $\Phi = 1$ on $x = 0$ for $0 < y < \pi/2$. Deduce the steady-state temperature distribution in a semi-infinite two-dimensional bar of width L, with the (infinitely) long sides held at zero temperature and short side held at temperature T_0.

10 The real parts of three analytic functions $f(x, y) = u(x, y) + iv(x, y)$ of $z = x + iy$ are

$$\sin x \cosh y, \qquad e^{y^2 - x^2} \cos 2xy, \qquad \text{and} \qquad \frac{x}{x^2 + y^2}$$

respectively. Use the Cauchy–Riemann equations to find their imaginary parts and hence deduce the forms of the complex functions.

Simplify your answers to become functions of z (rather than x and y separately). What do you notice about $u(x, 0)$ in each case?

Optional: by considering $\mathrm{d}f/\mathrm{d}z = \partial f/\partial x$ and writing this in terms of u only, deduce a slightly simpler procedure to find the analytic function f.

11 Let $f(z) = u(x, y) + iv(x, y)$ be an analytic function. Consider the geometric transformation from the xy-plane to the uv-plane defined by $(x, y) \mapsto (u(x, y), v(x, y))$. Find the Jacobian matrix J of this transformation. Use the Cauchy–Riemann equations to show that J can be expressed in the form

$$J = \begin{pmatrix} a & -b \\ b & a \end{pmatrix},$$

where $a, b \in \mathbb{R}$. Interpret what this matrix structure implies about the geometric interpretation of the local transformation.

Show that the families of level curves $u(x, y) = a$ and $v(x, y) = b$ (where a, b are constants) intersect orthogonally at all points where $f'(z) \neq 0$.

12 State the form the Laurent series for a function f with a pole of order n at the point z_0.

Consider the function

$$f(z) = \frac{1}{(z - 1)(z - 2)}.$$

Find the explicit Laurent series expansion of $f(z)$ centred at $z = 0$ valid for $|z| < 1$. Without calculating, how would you obtain the corresponding series valid for $1 < |z| < 2$ and $2 < |z|$?

13 Find the Laurent series expansion about $z = 0$ for the function:

$$f(z) = z^3 \cos\left(\frac{1}{z}\right).$$

Determine the region of convergence for this series. Classify the singularity at $z = 0$ and find the value of the coefficient of z^{-1}.

14⋆ The Casorati–Weierstrass theorem states that if z_0 is an isolated essential singularity of $f(z)$, then in any neighbourhood of z_0, the function $f(z)$ comes arbitrarily close to any complex number.

Illustrate this theorem using the function $f(z) = e^{1/z}$ at $z = 0$. Given any non-zero complex number $w = \rho e^{i\phi}$, show that there is an infinite sequence of points $z_n \to 0$ such that $f(z_n) = w$. What happens if $w = 0$?