Corrections to `ghm22@cam.ac.uk`. *Star* ($\star$) *indicates a harder question.*

1 Find the eigenvalues and eigenvectors of the matrix

$$\mathrm{A} = \begin{bmatrix} 1 & \alpha & 0 \\ \beta & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where neither of the complex constants α and β vanishes. Find the conditions for which (a) the eigenvalues are real, and (b) the eigenvectors are orthogonal. Hence show that both conditions are jointly satisfied if and only if A is Hermitian.

2 For a Hermitian matrix H, explain how to construct a unitary matrix U such that $\mathrm{U}^\dagger\mathrm{HU} = \mathrm{D}$, where D is a real diagonal matrix. Illustrate the procedure with the matrix

$$H = \begin{bmatrix} 4 & 3i \\ -3i & -4 \end{bmatrix}.$$

3 An anti-Hermitian matrix A is one for which $\mathrm{A}^\dagger = -\mathrm{A}$. What can be said about the eigenvalues of A?

If S is real symmetric and T is real antisymmetric, show that $\mathrm{T} \pm i\mathrm{S}$ are anti-Hermitian. Deduce that $\det(\mathrm{T} + i\mathrm{S} - \mathrm{I}) \neq 0$. Show that the matrix

$$\mathrm{U} = (\mathrm{I} + \mathrm{T} + i\mathrm{S})(\mathrm{I} - \mathrm{T} - i\mathrm{S})^{-1}$$

is unitary.

4 Let λ_i be the eigenvalues of an $n \times n$ Hermitian matrix A. Show that,

$$\mathrm{Tr}\,\mathrm{A}^k = \sum_{i=1}^{n} \lambda_i^k \quad \text{and} \quad \det \mathrm{A}^k = \prod_{i=1}^{n} \lambda_i^k,$$

and also that,

$$\det(\exp \mathrm{A}) = \exp(\mathrm{Tr}\,\mathrm{A}),$$

where $\exp \mathrm{A}$ is defined in the standard way,

$$\det \mathrm{A} = \sum_{n=0}^{\infty} \frac{A^n}{n!}.$$

Hence conclude that the exponential of an arbitrary real antisymmetric matrix acts as a rotation in $\mathbb{R}^n$.

5 Show that the eigenvalues of a real orthogonal matrix have unit modulus and that if λ is an eigenvalue then so is λ^*. Hence argue that the eigenvalues of a 3×3 real orthogonal matrix R must be a selection from $+1, -1$ and $e^{\pm i\alpha}$. Verify that $\det R = \pm 1$. What is the effect of R on vectors orthogonal to an eigenvector with eigenvalue ± 1?

6 Let H be an $n \times n$ Hermitian matrix with n distinct eigenvalues $\{\lambda_i\}$ and orthonormal eigenvectors $\{e_i\}$. Now consider the slightly perturbed matrix $H + \delta H$, where δH is small and

Hermitian. Let the eigenvalues and orthonormal eigenvectors of $H + \delta H$ be $\{\lambda_i + \delta\lambda_i\}$ and $\{e_i + \delta e_i\}$ respectively. By working to first order in the small quantities, show that

$$\delta\lambda_i = e_i^\dagger(\delta H)e_i, \quad \delta e_i = \sum_{j\neq i} \frac{e_j^\dagger(\delta H)e_i}{\lambda_i - \lambda_j} e_j.$$

Why is it permissible to omit any contribution to δe_i parallel to e_i?

[Hint: Write out the eigenvector equation and the orthonormality condition for the eigenvectors of the perturbed matrix. Expand, neglect products of small quantities, and simplify. Use the fact that $\{e_i\}$ is a basis.]

7 Find the eigenvalues and normalized eigenvectors of the symmetric matrices

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 0 & \sqrt{3} \\ 0 & 3 & 0 \\ \sqrt{3} & 0 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 1 & -3 \\ 3 & -3 & -3 \end{bmatrix}.$$

Describe the related quadratic surfaces.

8 A complex square matrix A is called *normal* if it commutes with its Hermitian conjugate, so that $\mathrm{AA}^\dagger = \mathrm{A}^\dagger\mathrm{A}$.

Prove that A is normal if and only if $|\mathrm{A}\mathbf{x}| = |\mathrm{A}^\dagger\mathbf{x}|$ for all vectors $\mathbf{x}$.

Show that if $\mathbf{x}$ is an eigenvector of a normal matrix A with eigenvalue λ, then $\mathbf{x}$ is also an eigenvector of $\mathrm{A}^\dagger$ with eigenvalue λ^*. Hence, prove that eigenvectors corresponding to distinct eigenvalues of a normal matrix must be mutually orthogonal.

9 Let H be an $n \times n$ Hermitian matrix with eigenvalues ordered such that $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. The Rayleigh quotient for a non-zero vector $\mathbf{x} \in \mathbb{C}^n$ is defined as:

$$R(\mathbf{x}) = \frac{\mathbf{x}^\dagger \mathrm{H} \mathbf{x}}{\mathbf{x}^\dagger \mathbf{x}}$$

By expanding $\mathbf{x}$ in terms of an orthonormal basis of eigenvectors of H, prove the Rayleigh–Ritz variational principle:

$$\lambda_1 \leq R(\mathbf{x}) \leq \lambda_n$$

Under what conditions regarding $\mathbf{x}$ do the upper and lower bounds hold with equality?

10 Let H be an $n \times n$ Hermitian matrix. Show that the matrix $(\mathrm{H} + i\mathrm{I})$ is always invertible.

The *Cayley transform* of H is defined as the matrix:

$$\mathrm{U} = (\mathrm{H} - i\mathrm{I})(\mathrm{H} + i\mathrm{I})^{-1}$$

Prove that U is a unitary matrix. Assuming that $(\mathrm{I} - \mathrm{U})$ is invertible, find an explicit expression for H in terms of U.

11 Let A be an invertible $n \times n$ complex matrix.

Show that $\mathrm{A}^\dagger\mathrm{A}$ is a Hermitian, positive-definite matrix. Let P be the unique positive-definite square root of $\mathrm{A}^\dagger\mathrm{A}$, such that $\mathrm{P}^2 = \mathrm{A}^\dagger\mathrm{A}$.

Show that the matrix $\mathrm{U} = \mathrm{AP}^{-1}$ is unitary. Deduce the Polar Decomposition: any invertible complex matrix can be uniquely written as the product of a unitary matrix and a positive-definite Hermitian matrix.