Corrections to `ghm22@cam.ac.uk`*. Star ($\star$) indicates a harder question.*

1 Use the Cauchy–Schwarz inequality and the properties of the inner product to prove the triangle inequality

$$|x+y| \leq |x| + |y|$$

for a complex vector space, where $|x|$ is the norm of the vector x. Under what conditions does equality hold?

2 Given a set of vectors $u_1, u_2, ..., u_m$ ($m \geq n$) that span an n-dimensional vector space, show that an orthogonal basis may be constructed by the Gram-Schmidt procedure:

$$e_1 = u_1$$

$$e_r = u_r - \sum_{s=1}^{r-1} \frac{e_s \cdot u_r}{e_s \cdot e_s} e_s \quad \text{for } r > 1.$$

What is the interpretation if any of the vectors e_r vanishes?

Find an orthonormal basis for the subspace of a four-dimensional Euclidean space spanned by the three vectors with components $(1,1,0,0)$, $(0,1,2,0)$ and $(0,0,3,4)$.

3 An $n \times n$ complex matrix A is such that each row and each column has exactly one non-zero element. The Hermitian conjugate of A is $A^\dagger = (A^T)^*$ (where A^T is the transpose of A, and A^* is its complex conjugate). Show that $A^\dagger A$ is a real diagonal matrix.

4$\star$ What does it mean to say that the vectors $\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_n$ are linearly independent?

Let A be a linear operator on an n-dimensional vector space, having n distinct eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$ and corresponding eigenvectors $e_1, e_2, ..., e_n$. Consider the action of the operator $A - \lambda_i I$ (where I is the identity operator) on the vector e_j in the cases $i = j$ and $i \neq j$.

Describe the action of this operator on $\sum_j \alpha_j \mathbf{e}_j$. Hence or otherwise show that the vectors $\{\mathbf{e}_i\}$ are linearly independent.

5 An $n \times n$ Hermitian matrix A is said to be positive definite if $\mathbf{x}^\dagger \mathrm{A} \mathbf{x} > 0$ for all non-zero vectors $\mathbf{x} \in \mathbb{C}^n$.

Show that if A is positive definite, then any principal submatrix (a square matrix formed by deleting the same set of rows and columns from A) is also positive definite. Deduce that all diagonal elements A_{ii} must be strictly positive real numbers.

For a 2×2 positive definite matrix A, prove that $\det \mathrm{A} \leq A_{11} A_{22}$. Under what condition does equality hold?

6 An Hermitian matrix A is one for which $A^\dagger = A$. Suppose that A and B are both Hermitian matrices. Show that $AB + BA$ is Hermitian. Also show that AB is Hermitian if and only if A and B commute.

7 Suppose A is an $n \times n$ matrix such that $\mathrm{A}^2 = \mathrm{A}$. By considering the eigenvalues of A show that either:

a. $\det \mathrm{A} = 1$ and $\mathrm{Tr}\, \mathrm{A} = n$; or,

b. $\det \mathrm{A} = 0$ and $\mathrm{Tr}\, \mathrm{A} = m$ with $m < n$.

Show that the matrix $\mathrm{B} = \mathrm{I} - \mathrm{A}$ also satisfies $\mathrm{B}^2 = \mathrm{B}$. Hence, show that any vector $\mathbf{x}$ can be uniquely decomposed as $\mathbf{x} = \mathbf{y} + \mathbf{z}$, where $\mathrm{A}\mathbf{y} = \mathbf{y}$ and $\mathrm{A}\mathbf{z} = \mathbf{0}$. What are the dimensions of the subspaces containing $\mathbf{y}$ and $\mathbf{z}$ in case (b)? Provide a geometric interpretation.

8 Show that two diagonalizable matrices A and B share a common basis of eigenvectors if and only if they commute. Use this result to deduce that the matrices $(\mathrm{I} + \mathrm{A})$ and $(\mathrm{I} - \mathrm{A})^{-1}$ commute.

9 An $n \times n$ complex matrix N is said to be nilpotent if $\mathrm{N}^k = 0$ for some positive integer k.

By considering the eigenvalue equation $\mathrm{N}\mathbf{x} = \lambda\mathbf{x}$ for a non-zero eigenvector $\mathbf{x}$, show that the only eigenvalue of a nilpotent matrix is $\lambda = 0$. Hence, determine $\det \mathrm{N}$ and $\mathrm{Tr}\, \mathrm{N}$.

Show that the matrix $(\mathrm{I} - \mathrm{N})$ is always invertible, and find an explicit expression for $(\mathrm{I} - \mathrm{N})^{-1}$ as a finite sum involving powers of N.

10 In quantum mechanics, the statistical state of a system is often represented by a *density matrix* ρ. A valid density matrix must be Hermitian, positive semi-definite ($\mathbf{x}^\dagger \rho \mathbf{x} \geq 0$ for all $\mathbf{x}$), and satisfy $\mathrm{Tr}\, \rho = 1$.

Show that the eigenvalues λ_i of ρ must satisfy $0 \leq \lambda_i \leq 1$.

By expressing the trace of ρ^2 in terms of these eigenvalues, prove the inequality:

$$\mathrm{Tr}\, \rho^2 \leq 1$$

Under what condition regarding the eigenvalues does $\mathrm{Tr}\, \rho^2 = 1$? (A state satisfying this condition is known as a *pure state*, whereas if $\mathrm{Tr}\, \rho^2 < 1$, it is a *mixed state*. You'll see this topic in Part II Physics, if you do it.)

11 Let H be an $n \times n$ Hermitian matrix representing the Hamiltonian (energy operator) of a quantum system. The time evolution of the state vector $\boldsymbol{\psi}(t) = U(t)\boldsymbol{\psi}(0)$ is governed by the matrix exponential $\mathrm{U}(t) = e^{-i\mathrm{H}t}$.

Using the power series definition of the matrix exponential,

$$e^{\mathrm{A}} = \sum_{n=0}^{\infty} \frac{A^n}{n!},$$

prove that $\mathrm{U}(t)$ is a unitary matrix.

If the initial state vector $\boldsymbol{\psi}(0)$ is normalized such that $|\boldsymbol{\psi}(0)|^2 = \boldsymbol{\psi}(0)^\dagger \boldsymbol{\psi}(0) = 1$, show that the state vector remains normalized at all times t.