

Corrections to ghm22@cam.ac.uk. Star (★) indicates a harder question.

Curvilinear coordinates are a new topic, but this sheet also has a few questions revising solutions of partial differential equations via separation of variables. This won't be new if you did course B last year. If you did course B, you may wish to skip those PDE questions in favour of the hard questions at the end.

- 1 Let q_i be orthogonal curvilinear coordinates. By requiring that $\nabla\phi \cdot d\mathbf{x} = \partial\phi/\partial x$ and applying the chain rule, show that,

$$\nabla\phi = \sum_i \frac{1}{h_i} \frac{\partial\phi}{\partial q_i} \mathbf{e}_i,$$

carefully defining the quantities h_i and $\mathbf{e}_i$.

- 2 Parabolic coordinates (u, v, ϕ) are defined in terms of Cartesian coordinates (x, y, z) by

$$x = uv \cos \phi, \quad y = uv \sin \phi, \quad z = \frac{1}{2}(u^2 - v^2)$$

Show that the surfaces of constant u , and those of constant v , are surfaces obtained by rotating parabolae about the z -axis. What are the surfaces of constant ϕ ? Show that the coordinate surfaces intersect at right angles and hence that these coordinates are orthogonal. Find the scale factors (h_u, h_v, h_ϕ) defined by

$$|d\mathbf{r}|^2 = h_u^2 du^2 + h_v^2 dv^2 + h_\phi^2 d\phi^2.$$

Hence obtain the Laplacian in these coordinates using the formula

$$\nabla^2 = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial q_1} \left(\frac{h_2 h_3}{h_1} \frac{\partial}{\partial q_1} \right) + \frac{\partial}{\partial q_2} \left(\frac{h_3 h_1}{h_2} \frac{\partial}{\partial q_2} \right) + \frac{\partial}{\partial q_3} \left(\frac{h_1 h_2}{h_3} \frac{\partial}{\partial q_3} \right) \right]$$

- 3 Elliptic cylindrical coordinates (u, v, z) are defined in terms of the Cartesian coordinates (x, y, z) by

$$x = a \cosh u \cos v,$$

$$y = a \sinh u \sin v,$$

$$z = z,$$

where $u \geq 0$, $0 \leq v < 2\pi$, $-\infty < z < \infty$. Sketch the surfaces of constant u and the surfaces of constant v . Show that the coordinate surfaces intersect at right angles and thus these coordinates are orthogonal coordinates.

Find the metric coefficients (h_u, h_v, h_z) defined by

$$dr^2 = dx^2 + dy^2 + dz^2 = h_u^2 du^2 + h_v^2 dv^2 + h_z^2 dz^2,$$

and show that $h_u = h_v$.

By calculating the relevant volume element $dV = h_u h_v h_z du dv dz$, evaluate the integral

$$I = \int_V (x^2 + y^2) dV$$

over the region V enclosed by the elliptic cylinder $u = U$ and the planes $z = 0$ and $z = H$.

- 4 Consider the two-stage transformation of Cartesian coordinates (x, y, z) given by

$$x = ax', \quad y = by', \quad z = cz'$$

$$x' = r' \sin \theta' \cos \phi', \quad y' = r' \sin \theta' \sin \phi', \quad z' = r' \cos \theta'$$

where a, b and c are positive constants. Calculate the Jacobian matrices of the transformations $(x, y, z) \mapsto (x', y', z')$ and $(x', y', z') \mapsto (r', \theta', \phi')$ and verify explicitly that

$$\frac{\partial(x, y, z)}{\partial(r', \theta', \phi')} = \frac{\partial(x, y, z)}{\partial(x', y', z')} \frac{\partial(x', y', z')}{\partial(r', \theta', \phi')}$$

Are the coordinates (r', θ', ϕ') orthogonal? What range of these coordinates is required to cover the interior of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1?$$

Express the volume element in coordinates (r', θ', ϕ') and hence calculate the volume of the ellipsoid.

- 5 Oblate spheroidal coordinates (ξ, η, ϕ) are related to Cartesian coordinates by

$$x = a \cosh \xi \sin \eta \cos \phi, \quad y = a \cosh \xi \sin \eta \sin \phi, \quad z = a \sinh \xi \cos \eta$$

where $\xi \geq 0$, $0 \leq \eta \leq \pi$, and $0 \leq \phi < 2\pi$.

Identify the geometric surfaces of constant ξ and constant η . Determine the metric scale factors h_ξ , h_η , and h_ϕ , and write down the expression for the gradient $\nabla \Psi$ in this coordinate system.

- 6 In a Cartesian coordinate system (x_1, x_2, x_3) , A is the point $(0, 0, -1)$, B is the point $(0, 0, 1)$ and P is an arbitrary point (x_1, x_2, x_3) . In a curvilinear coordinate system, the coordinates of P are specified by

$$u_1 = \frac{1}{2}(r_1 + r_2), \quad u_2 = \frac{1}{2}(r_1 - r_2), \quad u_3 = \phi$$

where r_1 and r_2 are the distances AP and BP respectively and ϕ is the angle between the planes ABP and $x_2 = 0$. Show that $x_3 = u_1 u_2$ and that the distance ρ from P to the x_3 -axis is equal to $\sqrt{(u_1^2 - 1)(1 - u_2^2)}$.

Next evaluate $\partial x_i / \partial u_j$ (with $i, j = 1, 2, 3$). Deduce that the curvilinear coordinates are orthogonal and sketch the coordinate surfaces. Show that the metric coefficients are

$$h_1 = \sqrt{\frac{u_1^2 - u_2^2}{u_1^2 - 1}}, \quad h_2 = \sqrt{\frac{u_1^2 - u_2^2}{1 - u_2^2}}, \quad h_3 = \sqrt{(u_1^2 - 1)(1 - u_2^2)}.$$

Show that if a function satisfies Laplace's equation and is independent of u_2 and u_3 then it has the form

$$\Psi = a + b \log \left(\frac{u_1 - 1}{u_1 + 1} \right)$$

for constants a and b .

- 7 Show that the solution of Laplace's equation, $\nabla^2 \Phi = 0$, in the region $0 < x < a$, $0 < y < b$, $0 < z < c$ with $\Phi = 1$ on the surface $z = 0$ and $\Phi = 0$ on the other surfaces, is

$$\Phi(x, y, z) = \frac{16}{\pi^2} \sum_{\text{odd } m} \sum_{\text{odd } n} \frac{\sin(m\pi x/a) \sin(n\pi y/b) \sinh(k(c-z))}{mn \sinh(kc)}$$

where $k^2 = m^2 \pi^2 / a^2 + n^2 \pi^2 / b^2$.

- 8 The temperature distribution $\theta(x, t)$ along a thin bar of length L satisfies the one-dimensional diffusion equation

$$\frac{\partial \theta}{\partial t} = \nu \frac{\partial^2 \theta}{\partial x^2}$$

where the diffusivity ν is a constant, t denotes time and x is the distance from one of the ends. Find $\theta(x, t)$ if the bar is insulated at each end (i.e. if $\partial\theta/\partial x = 0$ at each end), and if the initial temperature distribution is given by

$$\theta(x, 0) = 2\theta_0 \cos^2\left(\frac{\pi x}{L}\right),$$

where θ_0 is a constant. For large times what is the temperature distribution in the bar? Comment.

- 9★ Derive the expression for the divergence of a vector field $\mathbf{A} = \sum_i A_i \mathbf{e}_i$ in general orthogonal curvilinear coordinates:

$$\nabla \cdot \mathbf{A} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial q_1} (A_1 h_2 h_3) + \frac{\partial}{\partial q_2} (A_2 h_3 h_1) + \frac{\partial}{\partial q_3} (A_3 h_1 h_2) \right]$$

by considering the net flux of $\mathbf{A}$ out of an infinitesimal volume element $dV = h_1 h_2 h_3 dq_1 dq_2 dq_3$.

[Hint: If you're struggling with this, try it in Cartesian coordinates first.]

- 10★ Show that the $\mathbf{e}_1$ -component of the curl of a vector field $\mathbf{A}$ in orthogonal curvilinear coordinates is given by

$$(\nabla \times \mathbf{A})_1 = \frac{1}{h_2 h_3} \left[\frac{\partial}{\partial q_2} (h_3 A_3) - \frac{\partial}{\partial q_3} (h_2 A_2) \right]$$

by applying Stokes' Theorem ($\oint_C \mathbf{A} \cdot d\mathbf{x} = \iint_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S}$) to an infinitesimal rectangular loop lying within a surface of constant q_1 .

[Hint: As before, if struggling, try Cartesian first.]