

NST IB: Mathematical Methods I Sheet 2: Suffix notation & vector calculus

Corrections to *ghm22@cam.ac.uk*. Star (★) indicates a harder question.

This sheet should be largely revision from 1A, especially if you did the B course. Fluency with suffix notation and the summation convention is *incredibly* important for future topics in both this course and your later studies. If you're struggling after these questions, we can find more to practise.

- 1 In the following, the indices i, j, k, l, m take the values 1, 2, 3, and the summation convention applies. In particular, let $\mathbf{n}$ be a unit vector.

- a. Simplify the following expressions:

$$\begin{aligned} \delta_{ij}a_j, \quad \delta_{ij}\delta_{jk}, \quad \delta_{ij}\delta_{ji}, \quad \delta_{ij}n_in_j, \\ \epsilon_{ijk}\delta_{jk}, \quad \epsilon_{ijk}\epsilon_{ijl}, \quad \epsilon_{ijk}\epsilon_{ikj}, \quad \epsilon_{ijk}(a \times b)_k. \end{aligned}$$

- b. For each of the following equations, either give the equivalent in vector or matrix notation or explain why the equation is invalid:

$$\begin{aligned} x_i = a_ib_kc_k + d_i, \quad x_i = a_jb_i + c_kd_ie_kf_j, \quad u = \epsilon_{jkl}v_kw_lx_j, \\ \epsilon_{ijk}x_jy_k\epsilon_{ilm}x_ly_m = \mu, \quad A_{ik}B_{kl} = T_{ik}\delta_{kl}, \quad x_k = A_{ki}B_{ji}y_j. \end{aligned}$$

- c. Write the following equations in suffix notation using the summation convention:

$$\begin{aligned} (x + \mu y) \cdot (x - \mu y) = \kappa, \\ \mathbf{x} = |\mathbf{a}|^2\mathbf{b} - |\mathbf{b}|^2\mathbf{a}, \quad (2\mathbf{x} \times \mathbf{y}) \cdot (\mathbf{a} + \mathbf{b}) = \lambda. \end{aligned}$$

- d. Given that $A_{ij} = \epsilon_{ijk}a_k$ (for all i, j), show that $2a_k = \epsilon_{kij}A_{ij}$ (for all k).
 - e. Show that $\epsilon_{ijk}s_{ij} = 0$ (for all k) if and only if $s_{ij} = s_{ji}$ (for all i, j).
 - f. Given that $N_{ij} = \delta_{ij} - \epsilon_{ijk}n_k + n_in_j$ and $M_{ij} = \delta_{ij} + \epsilon_{ijk}n_k$, show that $N_{ij}M_{jk} = 2\delta_{ik}$.
- 2 Use suffix notation and the identity for $\epsilon_{ijk}\epsilon_{ilm}$ (which you should commit to memory) to prove the vector triple product identity:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}.$$

Use suffix notation to expand the curl of a cross product, $\nabla \times (\mathbf{F} \times \mathbf{G})$. Express your final answer in standard vector notation.

- 3 Let $\mathbf{r}$ be the position vector with components x_i , and let $r = |\mathbf{r}|$. Let $\mathbf{a}$ be a constant vector, T_{ij} be a constant symmetric tensor, and $f(r)$ be a differentiable scalar function.

- a. Using suffix notation, evaluate:

$$\frac{\partial r}{\partial x_i} \quad \text{and} \quad \frac{\partial^2 r}{\partial x_i \partial x_j}.$$

- b. Show that:

$$\nabla(\mathbf{a} \cdot \mathbf{r}) = \mathbf{a} \quad \text{and} \quad \nabla \cdot (f(r)\mathbf{r}) = 3f(r) + rf'(r).$$

- c. Define the scalar field $\Phi = T_{ij}x_ix_j$. Find an expression for the vector field $\mathbf{G} = \nabla\Phi$ in suffix notation, and show that $\nabla \cdot \mathbf{G} = 2T_{ii}$.

- d. Evaluate the components of the vector field $\mathbf{H} = \nabla \times (f(r)\mathbf{a} \times \mathbf{r})$ using suffix notation, and simplify your result into standard vector notation.

- 4 A fluid flow has the constant velocity vector (in Cartesian coordinates)

$$\mathbf{v}(\mathbf{r}) = (0, 0, W).$$

Explicitly calculate the volume flux of fluid,

$$Q = \int \mathbf{v} \cdot d\mathbf{S}$$

flowing across (a) the open hemispherical surface $r = a$, $z \geq 0$, and (b) the disc $r \leq a$, $z = 0$. Verify that the divergence theorem holds.

- 5 For a surface S enclosing a volume V , apply the divergence theorem to a vector field $\mathbf{F} = \mathbf{a}p$, where $\mathbf{a}$ is an arbitrary constant vector and $p(\mathbf{r})$ is a scalar field. Deduce that

$$\int_V \nabla p dV = \oint_S p d\mathbf{S}.$$

- 6 Consider the vector field $\mathbf{F}(\mathbf{r}) = \frac{\mathbf{r}}{r^3}$, where $\mathbf{r}$ is the position vector and $r = |\mathbf{r}|$. Show that $\nabla \cdot \mathbf{F} = 0$ everywhere except at the origin. Evaluate the surface integral $\oint_S \mathbf{F} \cdot d\mathbf{S}$ over the surface of a sphere of radius R centred at the origin. Is this result consistent with the divergence theorem?

- 7 A time-independent magnetic field $\mathbf{B}(\mathbf{r})$ is given by

$$\mathbf{B} = \frac{\mu_0 I}{2\pi} \frac{\mathbf{e}_z \times \mathbf{r}}{x^2 + y^2}$$

where μ_0 is the magnetic permeability and I is a constant. Using Cartesian coordinates, calculate the electric current density $\mathbf{J}$ given by the steady Maxwell equation $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$. Also evaluate $\oint_C \mathbf{B} \cdot d\mathbf{r}$, where C is a circle of radius a in the plane $z = 0$ and centred on $x = y = 0$. Discuss whether Stokes's theorem applies in this situation.

- 8 Show that, in Cartesian coordinates,

$$\nabla^2 \mathbf{F} = \nabla(\nabla \cdot \mathbf{F}) - \nabla \times (\nabla \times \mathbf{F}).$$

This vector identity remains true for all coordinate systems; however, for non-Cartesian coordinates,

$$(\nabla^2 \mathbf{F})_i \neq \nabla^2 F_i.$$

Why is this the case? Illustrate this point by evaluating $\nabla^2 \mathbf{F}$ for $\mathbf{F} = f(\rho)\mathbf{e}_\phi$ in cylindrical polar coordinates (ρ, ϕ, z) and comparing the result with $\nabla^2 f$.

- 9 Find the general circularly symmetric solution to the fourth-order equation

$$\nabla^4 \Psi \equiv \nabla^2(\nabla^2 \Psi) = 0.$$

Hint: use plane polar coordinates (ρ, ϕ) , and do not be too eager to expand everything out.

Find those circularly symmetric solutions in the unit disc that are equal to unity at the centre $\rho = 0$ and vanish on the boundary $\rho = 1$. Give a further condition to render the solution unique.

- 10 Let ϕ and ψ be scalar fields defined in a volume V bounded by a closed surface S . By applying the divergence theorem to the vector field $\mathbf{A} = \phi \nabla \psi$, prove Green's first identity,

$$\int_V (\phi \nabla^2 \psi + \nabla \phi \cdot \nabla \psi) \, dV = \oint_S \phi \nabla \psi \cdot d\mathbf{S}.$$

Hence, derive Green's second identity,

$$\int_V (\phi \nabla^2 \psi - \psi \nabla^2 \phi) \, dV = \oint_S (\phi \nabla \psi - \psi \nabla \phi) \cdot d\mathbf{S}.$$

We will use these later when we come to Green's function solutions to PDEs.