Corrections to `ghm22@cam.ac.uk`*. Star ($\star$) indicates a harder question.*

Fourier transforms are in a bit of an odd place in this course. We are doing some introductory work on them here in deference to the Physics courses, but we will revisit them again having covered the important topics on distributions (the Dirac delta function) and contour integration.

The definition of the Fourier transform sometimes differs between courses. Please assume that the definition is,

$$\tilde{f}(k) = \int_{-\infty}^{\infty} f(x)e^{-ikx}\,\mathrm{d}x\,,$$

with the inverse therefore being,

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(k)e^{ikx}\,\mathrm{d}k\,.$$

1 Let α and β be positive constants, and let H denote the Heaviside step function. Find the Fourier transforms of:

a. the odd function f_o defined for $x > 0$ by

$$f_o(x) = \begin{cases} 1, & 0 < x \leq 1, \\ 0, & x > 1. \end{cases}$$

b. the even function $f_e(x) = e^{-|x|}$.

c. the even function g, where

$$g(x) = \begin{cases} 1, & |x| < \alpha, \\ 0, & |x| \geq \alpha. \end{cases}$$

d. the function $h(x) = H(x)\sinh(\alpha x)e^{-\beta x}$, where $\alpha < \beta$.

2 Parseval's theorem is a special case of a more general identity. Suppose $f(x)$ and $g(x)$ have Fourier transforms $\tilde{f}(k)$ and $\tilde{g}(k)$ respectively. By expressing $f(x)$ and $g(x)$ in terms of their inverse Fourier transforms, show that

$$\int_{-\infty}^{\infty} f(x)g^*(x)\,\mathrm{d}x = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(k)\tilde{g}^*(k)\,\mathrm{d}k\,.$$

Verify that this reduces to the standard form of Parseval's theorem when $f(x) = g(x)$.

3

a. Use Parseval's theorem and the result of question 1a to show that

$$\int_{-\infty}^{\infty} \left(\frac{1-\cos x}{x}\right)^2 \mathrm{d}x = \pi$$

b. Use Parseval's theorem and the result of question 1b to evaluate the integral

$$\int_0^\infty \frac{\mathrm{d}k}{(1+k^2)^2}$$

[Note: Once you've done contour integration you'll be able to evaluate this directly.]

4 For g as given in question 1c, define

$$G(x) = \int_{-\infty}^{\infty} g(x-\xi)g(\xi)\,\mathrm{d}\xi\,.$$

By thinking about the graphical interpretation of convolution, find an expression for G. Explicitly demonstrate that the Fourier transforms of G and g satisfy the convolution theorem.

5 Show that:

a. if a function f and its Fourier transform $\tilde{f}$ are both real, then f is even;

b. if a function f is real and its Fourier transform $\tilde{f}$ is purely imaginary, then f is odd.

Are your answers to earlier questions consistent with these results?

6 Let $\tilde{f}(k)$ be the Fourier transform of $f(x)$. From the definition of the Fourier transform, prove the following properties:

a. the Fourier transform of $f(x-a)$ is $e^{-ika}\tilde{f}(k)$ for any real constant a;

b. the Fourier transform of $f(\lambda x)$ is $\frac{1}{|\lambda|}\tilde{f}\left(\frac{k}{\lambda}\right)$ for any non-zero real constant λ;

c. the Fourier transform of $f(x)\cos(ax)$ is $\frac{1}{2}\left[\tilde{f}(k-a)+\tilde{f}(k+a)\right]$.

7 Taking g as in question 1, for what values of α is the following integral zero for all k?

$$I(k) = \int_{-\infty}^{\infty} e^{20ik'}\tilde{g}(k')\tilde{g}(k-k')\,\mathrm{d}k'$$

8 By differentiating the definition of the Fourier transform with respect to k, show that the Fourier transform of $xf(x)$ is given by $i\frac{\mathrm{d}\tilde{f}}{\mathrm{d}k}$. Use this result, along with your answer to question 1b, to find the Fourier transform of $xe^{-|x|}$.

9 By taking the Fourier transform of the equation

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} - m^2\phi = f(x)$$

show that its solution $\phi(x)$ can be written as

$$\phi(x) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{ikx}\tilde{f}(k)}{k^2+m^2}\,\mathrm{d}k$$

where $\tilde{f}$ is the Fourier transform of f.

10 Consider the Gaussian function $f(x) = e^{-\alpha x^2}$, where $\alpha > 0$. We wish to find its Fourier transform.

By differentiating $f(x)$ with respect to x, show that $f'(x) + 2\alpha x f(x) = 0$. Take the Fourier transform of this differential equation to derive a first-order ordinary differential equation for $\tilde{f}(k)$. Solve this differential equation to find $\tilde{f}(k)$.

As α increases, the original Gaussian becomes narrower around the origin. What do you notice about its Fourier transform?

[Hint: You will need the standard integral $\int_{-\infty}^{\infty} e^{-\alpha x^2}\,\mathrm{d}x = \sqrt{\frac{\pi}{\alpha}}$ to find the constant of integration.]

11 Use the convolution theorem to solve the integral equation

$$\int_{-\infty}^{\infty} y(\xi)e^{-|x-\xi|}\,\mathrm{d}\xi = e^{-x^2}$$

for the unknown function $y(x)$. You may leave your answer in the form of an inverse Fourier transform, but you should simplify the integrand as much as possible.