

Corrections to *ghm22@cam.ac.uk*. Star (★) indicates a harder question.

This sheet builds on the last with more difficult contour integrals.

- 1 By considering the integral $\oint (z^2 + 1)^{-1} e^{ikz} dz$ taken around a large semicircle, show that for real positive k

$$\int_{-\infty}^{\infty} \frac{\cos(kx) dx}{x^2 + 1} = \pi e^{-k}$$

What is the value of the integral for $k \leq 0$?

- 2 Let n be an integer greater than 3. Deduce the required sector angle α for a wedge-shaped contour in the complex plane that allows for the direct evaluation of

$$\int_0^{\infty} \frac{dx}{1 + x^n}.$$

Show that the integral evaluates to $\frac{\pi}{n \sin(\pi/n)}$.

- 3 Let a be a real non-zero constant. Show that

$$\int_0^{\pi} \frac{d\theta}{a^2 + \sin^2 \theta} = \frac{\pi}{a\sqrt{1 + a^2}} \quad (a > 1).$$

- 4 By considering $f(z) = \exp(\frac{1}{2}iaz^2)$ and taking a wedge in the complex plane, show that

$$\int_0^{\infty} \cos\left(\frac{1}{2}ax^2\right) dx = \sqrt{\frac{\pi}{4|a|}}.$$

- 5 Choose appropriate branch cuts to show that:

a. $\int_0^{\infty} \frac{x^{-a} dx}{x + 1} = \frac{\pi}{\sin(\pi a)} \quad (0 < a < 1),$

b. $\int_0^{\infty} \frac{(\log x)^2 dx}{1 + x^2} = \frac{\pi^3}{8}.$

- 6 Evaluate the Dirichlet integral,

$$\int_0^{\infty} \frac{\sin x}{x} dx,$$

by considering $f(z) = \frac{e^{iz}}{z}$ integrated along a notched semicircular contour in the upper half plane with outer radius R and inner radius ϵ .

Note that this is an example of the *fractional residue theorem*: passing through an angle α counterclockwise around a simple pole yields a value of $i\alpha \operatorname{Res}(f, z_0)$.

[Hint: Evaluate the integral along the ϵ notch by directly parameterising it by $z = \epsilon e^{i\theta}$ with appropriate limits.]

- 7 By choosing an appropriate rectangular contour whose height is chosen to exploit the periodicity of the exponential function, evaluate the real integral

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1 + e^x} dx \quad (0 < a < 1).$$

Why does a standard semicircular contour fail for this integrand?

- 8 The reciprocal Gamma function can be written using Hankel's loop integral

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i} \int_C e^t t^{-z} dt$$

where C is an open contour starting at $-\infty - i\epsilon$, wrapping counterclockwise around the origin, and returning to $-\infty + i\epsilon$. For real $z < 1$, show how this open path completely evades the branch cut of t^{-z} , and use it to prove the reflection formula

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}.$$

- 9 By considering the integral of $(\cot z)/(z^2 + \pi^2 a^2)$ around a large square with vertices at $\pm(N + \frac{1}{2})\pi \pm i(N + \frac{1}{2})\pi$, prove that

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2 + a^2} = \frac{\pi}{a} \coth \pi a$$

provided that ia is not an integer. By considering a similar integral prove also that, if a is not an integer,

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+a)^2} = \frac{\pi^2}{\sin^2 \pi a}.$$

Find an expression for $\sum_{n=1}^{\infty} \frac{1}{n^2 + a^2}$ and take the limit as $a \rightarrow 0$ to deduce the value of $\sum_{n=1}^{\infty} \frac{1}{n^2}$.