

NST IB: Mathematical Methods II Sheet 6: Residues & Cauchy's theorem

Corrections to ghm22@cam.ac.uk. Star (★) indicates a harder question.

This sheet introduces you to the calculus of residues, Cauchy's theorem, and some relatively easy contour integration.

1 Establish the following general methods for calculating residues. [These are all very useful in practice and you are advised to memorize them.]

- a. If $f(z)$ has a simple pole at $z = z_0$ then the residue of f at $z = z_0$ ($\text{Res}(f, z_0)$) is given by $\lim_{z \rightarrow z_0} \{(z - z_0)f(z)\}$.
- b. If $f(z)$ is analytic then the residue of $f(z)/(z - z_0)$ at $z = z_0$ is $f(z_0)$.
- c. If $f(z)$ has a simple zero at $z = z_0$ then the residue of $1/f(z)$ at $z = z_0$ is $1/f'(z_0)$.
- d. If $h(z)$ has a simple zero at $z = z_0$ and $g(z)$ is analytic then the residue of $g(z)/h(z)$ at $z = z_0$ is $g(z_0)/h'(z_0)$.
- e. If $f(z)$ has a pole of order N at $z = z_0$ then the residue of $f(z)$ at $z = z_0$ is

$$\lim_{z \rightarrow z_0} \left\{ \frac{1}{(N-1)!} \frac{d^{N-1}}{dz^{N-1}} [(z - z_0)^N f(z)] \right\}.$$

2 Find the poles of the following functions and the residues at each pole:

- a. $\frac{z+1}{z^2}$
- b. $\frac{e^{-z}}{z^3}$
- c. $\frac{\sin^2 z}{z^5}$
- d. $\cot z$
- e. $\frac{z^2}{(1+z^2)^2}$

3 What is a branch point, and a branch cut? Describe how branch cuts can be used to make otherwise ill-defined contour integrals defined.

Sketch possible branch cuts for the following complex functions, giving the values on either side of each cut:

- a. $(z^2 + 1)^{\frac{1}{2}}$
- b. $(z^2 + 1)^{\frac{1}{3}}$
- c. $\log \left[\left(\frac{z-i}{z+i} \right)^2 \right]$

4 State and prove Cauchy's Theorem.

Let C be a closed contour that encloses, in a positive sense, the point z_0 in the complex z -plane, and let n be an integer. Show that

$$\oint_C (z - z_0)^n dz = \begin{cases} 2\pi i & n = -1 \\ 0 & n \neq -1. \end{cases}$$

Given that $f(z)$ is an analytic function, show that,

$$\oint_C \frac{f'(z) dz}{z - z_0} = \oint_C \frac{f(z) dz}{(z - z_0)^2}.$$

- 5 Let $f(z)$ be a function that is analytic within and on the circle $|z - z_0| = r$ in the z -plane. For non-negative integer n , show that

$$|f^{(n)}(z_0)| \leq \frac{n!}{r^n} \max_{|z-z_0|=r} |f(z)|.$$

Hence prove Liouville's Theorem: a function $f(z)$ that is analytic and bounded for all z is constant. Deduce that any polynomial $p(z)$ of degree at least one has at least one zero.

[Hint: consider $1/p(z)$.]

- 6 Let $f(z)$ be analytic within and on a circle C parametrized by $z = z_0 + re^{i\theta}$ for $0 \leq \theta \leq 2\pi$.

Use Cauchy's Integral Formula to show that

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta.$$

Hence, prove the maximum modulus principle: if $f(z)$ is non-constant and analytic on a bounded domain, $|f(z)|$ cannot attain a local maximum at an interior point.

- 7 Explain how the calculus of residues may be used to evaluate the integral of a function $f(z)$ around a closed contour C in the complex z -plane. Let

$$f(z) = \frac{z^n}{(z - a)(z - a^{-1})}$$

for real constant $a > 1$ and non-negative integer n . By choosing C to be a circle of unit radius, and using the calculus of residues, evaluate the integral

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{\cos(n\theta) d\theta}{1 - 2a \cos \theta + a^2}$$

- 8 By integrating around a rectangular contour with vertices at $\pm R$ and $i\pi \pm R$, where R is a large real constant, show that $\int_0^\infty \operatorname{sech} x dx = \pi/2$.
- 9 Let f be an analytic function inside and on a contour C . Assuming f is nonzero aside from an order n zero at z_0 , by considering the Laurent expansion of f about that zero, show that,

$$\frac{1}{2\pi i} \oint_C \frac{f'(z)}{f(z)} dz = n.$$

Let g be a meromorphic (analytic aside from a set of isolated poles) function inside and on C . Extend the argument above to show that,

$$\frac{1}{2\pi i} \oint_C \frac{g'(z)}{g(z)} dz = Z - P$$

where Z is the number of zeros and P is the number of poles of $g(z)$ inside C (counted with multiplicity).

- 10★ Let f be a function that is analytic everywhere in the complex plane \mathbb{C} except at a finite number of isolated poles z_1, z_2, \dots, z_n . Let C_R be a positively oriented (counterclockwise) circle centred at the origin with a radius R large enough to enclose all the poles.

We wish to define the idea of a *residue at infinity* by requiring that the contour integral C_R is the same whether we regard it as enclosing the poles at $\{z_i\}$ or enclosing the single pole at infinity.

Argue that from the perspective of the point at infinity, the path C_R is clockwise, and thus, from Cauchy's theorem,

$$\oint_{C_R} f(z) dz = -2\pi i \operatorname{Res}(f, \infty).$$

Make the substitution $z = 1/w$ to transform the point at infinity to zero. Does this change the orientation of the contour? Argue that the residue at infinity is,

$$\operatorname{Res}(f, \infty) = \operatorname{Res}\left(-\frac{1}{w^2} f\left(\frac{1}{w}\right), 0\right).$$