Corrections to `ghm22@cam.ac.uk`. *Star ($\star$) indicates a harder question.*

1 Let $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ be a set of orthonormal basis vectors. Confirm that the vectors

$$\mathbf{e}_1' = \frac{1}{2}(\mathbf{e}_1 + \sqrt{3}\mathbf{e}_3), \quad \mathbf{e}_2' = \mathbf{e}_2, \quad \mathbf{e}_3' = \frac{1}{2}(\mathbf{e}_3 - \sqrt{3}\mathbf{e}_1)$$

are orthonormal and hence define a new frame. If $\mathbf{v} = \sqrt{3}\mathbf{e}_1 + 5\mathbf{e}_2 + 2\mathbf{e}_3$, find its components, v_i' in the new frame. The moments of inertia of a rigid body, with respect to its principal axes $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, are 1, 2 and 3. Find the inertia tensor I' in the frame defined by $(\mathbf{e}_1', \mathbf{e}_2', \mathbf{e}_3')$. With respect to a third (orthonormal) frame $(\mathbf{e}_1'', \mathbf{e}_2'', \mathbf{e}_3'')$, the vector $\mathbf{e}_1$ has components $\frac{1}{\sqrt{2}}(1, 1, 0)$ and the vector $\mathbf{e}_2$ has components $\frac{1}{\sqrt{3}}(1, -1, 1)$. Find the inertia tensor I'' in this frame. Compute the trace of the square of I'' and verify that it agrees with the trace of the square of I; why must this be so? [The moments of inertia are the diagonal entries of the inertia tensor I in the frame provided by the (orthonormal) principal axes; the off-diagonal entries are zero in this frame.]

2 A right circular solid cylinder of uniform mass density ρ has radius a and length $\sqrt{3}a$. Identify the principal axes of this rigid body and in the frame defined by these axes find its inertia tensor about its centre of mass. Comment on your result.

3 The components of a vector $\mathbf{F}$ are related to those of vectors $\mathbf{J}$ and $\mathbf{H}$ by an equation of the form $F_i = T_{ijk}J_jH_k$ for arbitrary $\mathbf{J}$ and $\mathbf{H}$. Starting from the transformation law for the components of a vector, deduce that the coefficients T_{ijk} are the components of a 3rd-order tensor.

4 An isotropic tensor is defined as a tensor whose components are identical in all rotated coordinate systems. Prove that:

a. There are no non-zero isotropic first-order tensors (vectors).

b. The only isotropic second-order tensors are scalar multiples of the Kronecker delta δ_{ij}. [Hint: Consider the effect of a 90° rotation about one of the coordinate axes, and then a 45° rotation.]

5 Show that any second-order tensor T may be expressed, uniquely, as the sum of a symmetric tensor S with zero trace, an isotropic tensor I and an antisymmetric tensor A. Exhibit this decomposition for the tensor whose components are

$$\begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix} = \begin{pmatrix} 3 & -1 & 5 \\ 1 & 0 & 5 \\ 1 & -5 & 3 \end{pmatrix}$$

6 Let A be an antisymmetric 2nd-order tensor, with components A_{ij} in a frame in which the vector $\mathbf{x}$ has components x_i. Show that the vector $\mathbf{Ax}$, with components $A_{ij}x_j$, may be written as $\boldsymbol{\omega} \times \mathbf{x}$ for some vector $\boldsymbol{\omega}$ determined by A. Now show that the tensor $B = A^2$ is symmetric, and find its eigenvalues in terms of $\boldsymbol{\omega}$.

7 The electrical conductivity tensor σ_{ij} determines the electric current density $\mathbf{J}$ that flows in response to a uniform and constant applied electric field $\mathbf{E}$, according to the formula $J_i = \sigma_{ij}E_j$. The electrical conductivity tensor of a crystal is measured to have entries

$$\sigma_{ij} = \begin{pmatrix} 1 & \sqrt{2} & 0 \\ \sqrt{2} & 3 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

in a laboratory frame. Show that there is a direction in which no current flows, and find that direction. The rate of energy dissipation per unit volume is given by $\mathbf{E} \cdot \mathbf{J}$. For $|\mathbf{E}|^2 = 1$ find the minimum and maximum possible energy dissipation rates.

8 The eigenvalues of the conductivity tensor σ_{ij} of an anisotropic crystal are α, β and γ. Write down the tensor in the frame in which it is diagonal and identify the axis of symmetry due to a repeated eigenvalue (e.g., if $\alpha = \beta$). A particular crystal is grown such that it takes the form of a long thin wire of length L and cross-sectional area A, the direction of the wire making an angle θ with the axis of symmetry of the crystal. Show that the wire's resistance to a current passing along it is

$$R = (L/A)(\alpha^{-1} \cos^2 \theta + \gamma^{-1} \sin^2 \theta).$$

9 Let M be a second-order tensor with components M_{ij} in one frame and components M'_{ij} in another frame. Explain (i) why $\epsilon_{ijk} \det M = \epsilon_{lmn} M_{il} M_{jm} M_{kn}$ and (ii) why ϵ_{ijk} are the components of a 3rd-order isotropic pseudo-tensor. (iii) Deduce that $\det M' = \det M$, and hence that $\det M$ is a scalar.

10 A fourth-order tensor has components

$$c_{ijkl} = \alpha\delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk},$$

where α, β and γ are constants. Show that this defines an isotropic tensor. It is, in fact, the most general fourth-order isotropic tensor, and you may assume this for the rest of the question. In an isotropic fluid the (scalar) pressure p and the second-order stress tensor σ are related by $p = -\frac{1}{3}\sigma_{ii}$. Given that the deviatoric stress tensor s is defined by $s_{ij} = \sigma_{ij} + p\delta_{ij}$, show that it has zero trace. For a fluid with velocity field $\mathbf{v}(\mathbf{x})$, the second-order strain tensor e is defined by

$$e_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$$

In a "Newtonian" fluid the deviatoric stress tensor s is linearly related to the strain tensor e by a fourth-order tensor. Show that the relationship between $\sigma ande$ must take the form

$$\sigma_{ij} = 2\mu\left(e_{ij} - \frac{1}{3}\delta_{ij}e_{kk}\right) - p\delta_{ij}$$

for some constant μ (known as the coefficient of viscosity). Given that $\mu > 0$, show further (by considering an appropriate frame, or otherwise) that the scalar $s_{ij}e_{ij}$ is non-negative.

11 A second-order tensor T is defined by $T_{ij} = \delta_{ij} + \epsilon_{ijk}x_k$ where x_k are the (Cartesian) components of a position vector $\mathbf{x}$. Compute the following surface integrals, where in each case the surface of integration is the unit sphere $|\mathbf{x}| = 1$:

a. $\iint x_i \, \mathrm{d}S$,

b. $\iint T_{ij} \, \mathrm{d}S$,

c. $\iint T_{ij}T_{jk} \, \mathrm{d}S$,

d. $\iint n_i n_j n_k n_l \, \mathrm{d}S$ where n is a unit vector.

12⋆ State the Divergence Theorem in suffix notation for a second-order tensor field $\sigma_{ij}(\mathbf{x})$ defined over a volume V bounded by a closed surface S with outward unit normal n_i.

A continuous medium occupying volume V is in static equilibrium under the action of a surface traction vector $t_i = \sigma_{ij} n_j$ and a body force per unit volume f_i, meaning that $\frac{\partial \sigma_{ij}}{\partial x_j} + f_i = 0$ throughout V. The total torque about the origin exerted on the body is given by $T_i = \int_S \epsilon_{ijk} x_j t_k \, \mathrm{d}S + \int_V \epsilon_{ijk} x_j f_k \, \mathrm{d}V$. Show that this total torque can be expressed as

$$T_i = \int_V \epsilon_{ijk} \sigma_{kj} \, \mathrm{d}V \, .$$

Deduce that local conservation of angular momentum requires the stress tensor σ_{ij} to be symmetric.

13⋆ In an electrostatic field with electric vector E_i, the Maxwell stress tensor is defined by

$$\Sigma_{ij} = \epsilon_0 \left(E_i E_j - \frac{1}{2} E_k E_k \delta_{ij} \right) ,$$

where ϵ_0 is a constant scalar.

(a) Show that the divergence of this tensor can be written as

$$\frac{\partial \Sigma_{ij}}{\partial x_j} = \epsilon_0 \left(E_i \frac{\partial E_j}{\partial x_j} + E_j \left(\frac{\partial E_i}{\partial x_j} - \frac{\partial E_j}{\partial x_i} \right) \right) .$$

(b) In a region containing a local charge density ρ but no time-varying magnetic fields, the governing Maxwell equations are $\frac{\partial E_j}{\partial x_j} = \frac{\rho}{\epsilon_0}$ and $\epsilon_{jkl} \frac{\partial E_l}{\partial x_k} = 0$. Deduce that $\frac{\partial \Sigma_{ij}}{\partial x_j} = \rho E_i$. What is the physical interpretation of this result?