

Corrections to *ghm22@cam.ac.uk*. Star (★) indicates a harder question.

- 1 The quarter-plane  $x > 0$ ,  $y > 0$  is occupied by a single source of heat of strength  $Q$  positioned at the point  $(x_0, y_0)$ . At  $x = 0$  there is a plane conducting wall held at temperature  $T_0$ . At  $y = 0$  there is an insulated wall across which no heat can flow (i.e. the heat flux normal to the wall must vanish). Write down the equation and boundary conditions satisfied by a time-independent temperature field  $T(x, y)$  and use the method of images to find it. Hence show that the magnitude of the heat flux across the wall at the point  $(0, y)$  is

$$\frac{Qx_0}{\pi} \left\{ \frac{1}{x_0^2 + (y - y_0)^2} + \frac{1}{x_0^2 + (y + y_0)^2} \right\}.$$

Calculate the total heat radiated across the wall at  $x = 0$  and comment on the result.

- 2 Define the Green's function  $G(\mathbf{r}, \mathbf{r}')$  for the three-dimensional Laplacian with Dirichlet boundary conditions, acting on functions defined in a volume  $V$  with bounding surface  $S$ . Given that  $\nabla^2 u = 0$  in  $V$  and that  $u = f$  on  $S$ , show that

$$u(\mathbf{r}') = \int_S f(\mathbf{r}) \frac{\partial G}{\partial n} dS$$

where  $\partial/\partial n$  denotes differentiation along the outward normal on  $S$ . What is the analogous equation for two space dimensions? Show that a solution of  $\nabla_{\mathbf{r}}^2 G(\mathbf{r}, \mathbf{r}') = \delta^{(3)}(\mathbf{r} - \mathbf{r}')$  in two dimensions is  $G(\mathbf{r}, \mathbf{r}') = \frac{1}{2\pi} \log |\mathbf{r} - \mathbf{r}'|$ . Hence find the Green's function, with Dirichlet boundary conditions, for the two-dimensional Laplacian in the half-plane  $-\infty < x < \infty$ ,  $y > 0$ . Use it to solve the Laplace equation for  $u$  in this region with boundary conditions that  $u \rightarrow 0$  as  $r \rightarrow \infty$ ,  $u = 0$  on  $y = 0$  for  $|x| > 1$ , and  $u = 1$  on  $y = 0$  for  $|x| \leq 1$ .

- 3 A thin disc of uniform density and total mass  $M$  lies in the plane  $z = 0$  of cylindrical polar coordinates  $(r, \phi, z)$  and occupies the region  $r \leq a$  of this plane. Use the integral expression for a solution of Poisson's equation to find the gravitational potential  $\Phi(r, z)$  on the axis of symmetry  $r = 0$ . Assuming that  $|z| \gg a$ , expand your result as a power series accurate to  $\mathcal{O}(z^{-5})$ . Now write down the general solution of Laplace's equation in spherical polar coordinates valid at large distance  $r_s$  from the origin. By comparing this with your previous expansion, find an expression for  $\Phi(r, \theta)$  that is valid off the axis of symmetry. [Hint: Recall that  $P_n(\cos \theta) = 1$  when  $\cos \theta = 1$ .]
- 4 State a version of Green's identity applicable to a plane surface  $S$  bounded by a closed curve  $C$ . Use this identity, and the Green's function from Question ??, to show that the solution of  $\nabla^2 \Phi = 0$  within the disc of radius  $a$ , i.e.  $r < a$  in plane polar coordinates  $(r, \phi)$ , subject to the boundary condition that  $\Phi = \Psi(\phi)$  on the boundary at  $r = a$ , is

$$\Phi(r, \phi) = \frac{a^2 - r^2}{2\pi} \int_0^{2\pi} \frac{\Psi(\phi') d\phi'}{a^2 - 2ar \cos(\phi - \phi') + r^2}.$$

- 5 Use the result of Question 4 to show that in  $0 \leq r < 1$ , and for  $F(r, \phi, \phi') \equiv 1 - 2r \cos(\phi - \phi') + r^2$ :

$$\begin{aligned} \int_0^{2\pi} \frac{d\phi'}{F(r, \phi, \phi')} &= \frac{2\pi}{1 - r^2} \\ \int_0^{2\pi} \frac{\sin(\phi') d\phi'}{F(r, \phi, \phi')} &= \frac{2\pi r \sin \phi}{1 - r^2} \\ \int_0^{2\pi} \frac{\cos^2(\phi') d\phi'}{F(r, \phi, \phi')} &= \frac{2\pi r^2 \cos^2 \phi}{1 - r^2} + \pi. \end{aligned}$$

- 6 Find the spherically symmetric, free-space Green's function  $G(\mathbf{r}, \mathbf{r}')$  for the three-dimensional Helmholtz equation

$$(\nabla^2 + k^2)G(\mathbf{r}, \mathbf{r}') = \delta^{(3)}(\mathbf{r} - \mathbf{r}').$$

By substituting  $G = f(R)/R$ , where  $R = |\mathbf{r} - \mathbf{r}'|$ , and keeping only the solution proportional to  $e^{ikR}$ , show that

$$G(\mathbf{r}, \mathbf{r}') = -\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}.$$

[Hint: Consider integrating the equation over a small sphere centred at  $R = 0$  to fix the normalization constant.]

- 7 Consider the Helmholtz equation  $(\nabla^2 + k^2)u = 0$  in the half-space  $z > 0$ , subject to the inhomogeneous Dirichlet boundary condition  $u(x, y, 0) = f(x, y)$  on the plane  $z = 0$ . Use the method of images and the free-space Green's function from the previous question to construct the appropriate Dirichlet Green's function  $G_D(\mathbf{r}, \mathbf{r}')$  for this domain. Hence, show that the solution can be written in the form

$$u(\mathbf{r}') = \frac{1}{2\pi} \iint_{-\infty}^{\infty} f(x, y) \left( \frac{ik}{R} - \frac{1}{R^2} \right) \frac{z'}{R} e^{ikR} dx dy,$$

where  $R^2 = (x - x')^2 + (y - y')^2 + z'^2$ .

- 8 Find the Green's function for the Laplacian in the infinite two-dimensional strip  $-\infty < x < \infty$ ,  $0 < y < L$ , subject to Dirichlet boundary conditions  $G = 0$  on the parallel walls  $y = 0$  and  $y = L$ . [Hint: You may find it helpful to use a Fourier transform in the  $x$ -direction while maintaining an explicit differential structure in the  $y$ -direction.]