*Corrections to* `ghm22@cam.ac.uk`. *Star ($\star$) indicates a harder question.*

1 Consider transformations $z \mapsto az + b$ acting on complex numbers $z$ where $a$ and $b$ are complex and $|a| = 1$. Show that these transformations form a group. Show that the $2 \times 2$ complex matrices

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$$

form a faithful representation. Show that, within the 2-dimensional space of column vectors with two entries, there is an invariant subspace.

2 Let $\Sigma_3$ be the permutation group on 3 objects. Show that $|\Sigma_3| = 6$ and that $\Sigma_3$ is isomorphic to $D_3$, the symmetry group of an equilateral triangle. By considering the action of $\Sigma_3$ in permuting the components of a vector $(a, b, c)^T$ in 3-dimensional space $\mathbb{R}^3$, or otherwise, show that the matrices,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

provides a 3-dimensional faithful representation of $\Sigma_3$. Explain the statement that the matrices are the representatives of the permutations (23), (31), (12), (123) and (132) in the order shown.

Show that the representation is reducible by verifying that the vectors $(a, a, a)^T$ form a 1-dimensional invariant subspace. Find a further 2-dimensional invariant subspace.

Consider the matrix,

$$S = \begin{pmatrix} \alpha & 0 & 2\beta \\ \alpha & \beta\sqrt{3} & -\beta \\ \alpha & -\beta\sqrt{3} & -\beta \end{pmatrix},$$

for nonzero $\alpha$ and $\beta$, and $E$ representing the matrices above, the matrix products $S^{-1}ES$ are block-diagonal. How are these transformed matrices related to the irreducible representations of $\Sigma_3$? What goes wrong if $\alpha = 0$ or $\beta = 0$?

3 Consider the following mappings from $D_4$ (the symmetry group of a square) into or onto $C_2$ with $C_2$ represented as $\{1, -1\}$:

$$\phi_1 : \{I, R, R^2, R^3, m_1, m_2, m_3, m_4\} \mapsto \{1, 1, 1, 1, -1, -1, -1, -1\}$$
$$\phi_2 : \{I, R, R^2, R^3, m_1, m_2, m_3, m_4\} \mapsto \{1, -1, 1, -1, -1, -1, 1, 1\}$$
$$\phi_3 : \{I, R, R^2, R^3, m_1, m_2, m_3, m_4\} \mapsto \{1, -1, 1, -1, 1, -1, -1, 1\}$$

in the order displayed. Show that the first two are homomorphisms but that the last is not. Verify that the kernels of the first two mappings are all normal subgroups of $D_4$ and that the kernel of the last mapping is not.

4 Let $G$ be an Abelian group with $|G|$ elements. Show that each element of $G$ forms a conjugacy class by itself. Deduce that there are $|G|$ one-dimensional representations of $G$ and no other irreducible representations. Find all the one-dimensional representations of the cyclic group $C_n$.

5 List the five conjugacy classes of $D_4$, the symmetry group of the square. Calculate the character table of $D_4$, and state the properties that you can use to check your calculation.

6 Let $e_1$ and $e_2$ be unit vectors in the plane separated by an angle of $120°$, $\Delta$ the equilateral triangle with vertices $e_1$, $e_2$ and $e_3 = -(e_1 + e_2)$ and $D_3$ the symmetry group of $\Delta$. Calculate the matrices of the two-dimensional irreducible representation of $D_3$ by considering the action on vectors in the plane, taking $e_1$ and $e_2$ as basis vectors. Show that the traces of these matrices agree with those in the character table of $D_3$. Verify that the orthogonality theorem is satisfied, i.e. that

$$\sum_g d^{(\alpha)}(g)_{ij} d^{(\alpha)}(g^{-1})_{kl} = \frac{|G|}{n_\alpha} \delta_{il} \delta_{jk}$$

7 Let $D$ be a unitary representation of a finite group $G$ and $\{\chi(g) : g \in G\}$ the character of $D$. Show that

$$\frac{1}{|G|} \sum_g \chi(g)^* \chi(g)$$

is a positive integer, equal to 1 if and only if $D$ is irreducible.

8 Let $D_1 : G \to GL(n, \mathbb{C})$ be a representation of a group $G$. (Recall $GL(n, \mathbb{C})$ is the group of $n \times n$ invertible complex matrices.) Define $D_2(g) = [D_1(g^{-1})]^\dagger$. Show that $D_2$ is a representation.

Suppose that $W$ is an invariant subspace of $\mathbb{C}^n$ with respect to $D_2$. Let $W^\perp$ be the vector space of vectors orthogonal to $W$, and show that $W^\perp$ is an invariant subspace of $\mathbb{C}^n$ with respect to $D_1$. Finally show that if $D_1$ is irreducible, then $D_2$ must also be irreducible.

9 Let $D(g)$ be a representation of a finite group $G$ on a complex vector space $V$. Let $\langle \cdot, \cdot \rangle$ be an arbitrary inner product defined on $V$. Define a new inner product $[\cdot, \cdot]$ by summing over the group actions:

$$[x, y] = \frac{1}{|G|} \sum_{g \in G} \langle D(g)x, D(g)y \rangle$$

Show that $[x, y]$ is a group-invariant inner product, meaning $[D(h)x, D(h)y] = [x, y]$ for all $h \in G$. Prove that if a basis is chosen for $V$ that is orthonormal with respect to this group-invariant inner product, then the matrices $D(g)$ are unitary for all $g \in G$.

10 Obtain the character table for $D_3$, the symmetries of a triangle.

Verify the column orthogonality theorem for this character table, which states that for any two conjugacy classes $C_i$ and $C_j$,

$$\sum_\alpha \chi^{(\alpha)}(C_i) \chi^{(\alpha)}(C_j)^* = \frac{|G|}{|C_i|} \delta_{ij}$$

where the sum runs over all inequivalent irreducible representations $\alpha$.

11 Consider the dihedral group $D_4$, the symmetry group of a square with vertices labeled $1, 2, 3, 4$ in counter-clockwise order. By considering the action of the group elements in permuting these four vertices, write down the character $\chi_V$ of the corresponding 4-dimensional permutation representation.

Using the character table of $D_4$:

| | $I$ | $R^2$ | $2R$ | $2m$ | $2m'$ |
|---|---|---|---|---|---|
| $\chi^{(1)}$ | 1 | 1 | 1 | 1 | 1 |
| $\chi^{(2)}$ | 1 | 1 | 1 | $-1$ | $-1$ |
| $\chi^{(3)}$ | 1 | 1 | $-1$ | 1 | $-1$ |
| $\chi^{(4)}$ | 1 | 1 | $-1$ | $-1$ | 1 |
| $\chi^{(5)}$ | 2 | $-2$ | 0 | 0 | 0 |

determine how this 4-dimensional representation decomposes into a direct sum of irreducible representations. Identify the 1-dimensional invariant subspace corresponding to $\chi^{(1)}$ explicitly in terms of the components of a vector $(x_1, x_2, x_3, x_4)^T \in \mathbb{C}^4$.