Corrections to `ghm22@cam.ac.uk`*. Star* ($\star$) *indicates a harder question.*

1 $GL(n, \mathbb{R})$ is the group of all invertible $n \times n$ real matrices (with matrix multiplication as the group product operation), and $\mathbb{R}^*$ denotes the nonzero real numbers, which form a group under multiplication.

Show that the map $\Phi : GL(n, \mathbb{R}) \mapsto \mathbb{R}^*$ defined by $\Phi(M) = \det(M)$ is a homomorphism. What is the kernel of Φ? Show that the kernel is a normal subgroup of $GL(n, \mathbb{R})$ and describe its cosets. Show that the product of two cosets is well-defined and produces another coset. Deduce that the set of all cosets forms a group: this is called the quotient group $GL(n, \mathbb{R})/\ker(\Phi)$.

What group is $GL(n, \mathbb{R})/\ker(\Phi)$ isomorphic to?

2 What is the order of a group G and of an element $g \in G$?

Given two groups G and H, the *direct product* $G \times H$ is defined as Cartesian product of the two underlying sets (that is, all ordered elements (g, h) for each $g \in G$, $h \in H$) imbued with the group operation,

$$(g_1, h_1) \cdot (g_2, h_2) = (g_1 g_2, h_1 h_2).$$

Show that the direct product forms a group. What is the order of $G \times H$ in terms of the orders of G and H? What about the order of (g, h) in terms of the orders of g and h?

Show that $C_2 \times C_3$ is isomorphic to C_6 but that $C_2 \times C_4$ is not isomorphic to C_8. In general, when is $C_n \times C_m$ isomorphic to C_{nm}?

3 State and prove Lagrange's theorem. Prove the corollary that the order of a group element divides the order of its group.

Prove that if $|G| = p$ is prime then G is isomorphic to C_p.

Prove that if G has elements only of order 1 or 2, then G is abelian. Hence or otherwise, prove that if $|G| = 2p$ (p prime), then G must contain an element of order p.

4 The *index*, $[G : H]$, of a subgroup H of a group G is the number of cosets of H in G. Suppose that $[G : H] = 2$. Prove that H must be a normal subgroup of G. Give an example of a subgroup of index 3 in a finite group that is not normal.

5 The centre of a group G, denoted $Z(G)$, is defined as the set of elements that commute with all elements of G:

$$Z(G) = \{g \in G \mid gx = xg \text{ for all } x \in G\}.$$

Prove that $Z(G)$ is a normal subgroup of G. Furthermore, prove that if the quotient group $G/Z(G)$ is cyclic, then G must be Abelian.

6 Let H and K be subgroups of a group G. Prove that the intersection $H \cap K$ is always a subgroup of G. If both H and K are normal in G, prove that $H \cap K$ is also normal in G.

Define the subset $HK = \{hk \mid h \in H, k \in K\}$. Show by a counterexample (e.g., using Σ_3) that HK is not necessarily a subgroup of G. Prove that HK is a subgroup if and only if $HK = KH$.

7 The Pauli spin matrices used in quantum mechanics are defined as:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Let G be the set of 16 matrices defined by multiplying each element of $\{\sigma_x, \sigma_y, \sigma_z\}$ by $\{\pm I, \pm i\}$.

a. Show that G forms a non-Abelian group under matrix multiplication (this is known as the Pauli group).

b. Find the centre $Z(G)$ of this group and determine its order.

c. Identify the structure of the quotient group $G/Z(G)$ and show that it is isomorphic to the Klein four-group V.

8 Let $\phi : G \to H$ be a group homomorphism. Prove that if $g \in G$ has finite order n, then the order of its image $\phi(g)$ in H must divide n. Show that if ϕ is an isomorphism, then the order of $\phi(g)$ is exactly equal to n.

Hence or otherwise prove that the additive group of real numbers $(\mathbb{R}, +)$ cannot be isomorphic to the multiplicative group of non-zero real numbers $(\mathbb{R}^*, \cdot)$.

9 Let p be a prime number. Consider the set G of all 3×3 upper-triangular matrices over the integers modulo p, $\mathbb{Z}_p$, with 1s on the main diagonal:

$$M = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \quad \text{where } a, b, c \in \mathbb{Z}_p.$$

Prove that G forms a non-Abelian group under matrix multiplication (this is the discrete Heisenberg group over $\mathbb{Z}_p$). What is its order?

For the specific case $p = 3$, calculate M^3 for an arbitrary element. Show that every non-identity element in this group has order 3.

10⋆ For any element g in a group G, the *centralizer* of g is defined as the set of elements in G that commute with g:

$$C_G(g) = \{x \in G \mid xg = gx\}$$

a. Prove that $C_G(g)$ is a subgroup of G. Is it necessarily a normal subgroup?

b. Let $\{g\} = \{xgx^{-1} \mid x \in G\}$ be the conjugacy class of g. Establish a bijection between the set of left cosets of $C_G(g)$ and the elements of $\{g\}$.

c. Deduce that for a finite group G, the number of elements in the conjugacy class of g must precisely divide the order of G.